Algebraic geometry 2

Exercise Sheet 3

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Exercise 1. Let A be an integral domain. Show that the topological space $\operatorname{Spec} A$ (with Zariski topology) is irreducible.

Exercise 2. Let A be a ring and consider the topological space $\operatorname{Spec} A$ (with Zariski topology).
(1) Show that $\operatorname{Spec} A$ is not connected if and only if there exists a non-trivial idempotent $e \in A$ (that is $e^2 = e$ and $e \neq 0, 1$).
(2) Show that $\operatorname{Spec} A$ is not connected if and only if $A \simeq A_1 \times A_2$, where A_1 and A_2 are non-zero rings.
Hint: Follows from (1).
(3) Let A be a local ring. Show that $\operatorname{Spec} A$ is connected.
Hint: Use (1).

Exercise 3. Let A be a ring and $S \subset A$ a multiplicatively closed subset.
(1) Recall the universal property of the localization $S^{-1}A$.
(2) Let $f, h \in A$ and assume that $D(h) \subset D(f)$ in $\operatorname{Spec} A$. Let $i : A \to A_h$ and $j : A \to A_f$ be the respective canonical ring homomorphisms.
Using the universal property show that there exists a unique ring homomorphism $\varphi : A_f \to A_h$, such that $\varphi \circ j = i$.
(3) Give an interpretation of 2) in terms of the structural sheaf $\mathcal{O}$ on $\operatorname{Spec} A$.

Exercise 4. (1) Let $\mathcal{F}$ be a presheaf on a topological space X, let $\mathcal{B}$ be a basis of the topology on X and $x \in X$.
Show that
$$\mathcal{F}_x \simeq \varinjlim_{x \in U \in \mathcal{B}} \mathcal{F}(U)$$
(2) Let A be a ring and $\rho \in \operatorname{Spec} A$. Show that
$$\varinjlim_{\rho \in D(f)} A_f \simeq A_\rho \,.$$
Remark: Give two proofs of (2): one using only definitions (commutative algebra) and another one using Proposition 2.10 from the lecture.